A Method of Developing the Perturbative Function of Planetary Motion.

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THE development of the perturbative function in powers of the eccentricities and inclinations has of late been generally regarded as of little value, owing to the complex character of the series to which it leads. It is, in consequence, but little used, even in those cases of nearly circular orbits where it would be most convenient. Still, it is the only development in which the disturbing force is given as an explicit function of all the elements, and is therefore of more interest to the geometer than any other. Moreover, it admits of various simplifications in its application to the numerical problems of celestial mechanics which deserve more attention than they have received, and which may entitle it to a more favorable comparison with other methods than it has been supposed to offer.

The object of the present paper is to exhibit a method of effecting the development in powers of the eccentricities, which seems to me to offer some features of interest, and possibly to contain the germ of some principle which I have not fully grasped, and which may admit of wider and more important applications. I refer especially to the expression of the coefficient of each power of the eccentricity in terms of the coefficients of lower powers, and to the expression of the coefficient of each term involving the perihelia of two planets as the symbolic product of coefficients involving the perihelion of one only. The first of these features was pointed out in a note to the French Academy, found in the *Comptes Rendus*, Vol. LXX. p. 385, the ground of which is covered by the present paper. The second was discovered on the completion of the theory some years later.

One great practical advantage of the process is that it is reduced to a uniform operation of algebraic multiplication, which can be executed by an

unskilled computer, and can be carried to any extent without repeating the previous processes. I see no reason why it might not be possible to express the terms of the moon's longitude by some similar series of operations, and thus greatly simplify the practical problem of the lunar theory. Probably each step would be found to involve the solution of a differential equation, but this equation might be of a very simple character, and only a particular integral would be required.

The present development offers nothing new in the method of taking account of the mutual inclination of the orbits. The development with respect to this element may be made by any method which gives the coefficients as known functions of the radii vectores and the mutual inclination, and the elements under the signs sine and cosine are the multiples of the distances of the two planets from the common node.

The expression to be developed is

$$R = (r^2 - 2\,rr' \cos V - r'^2)^{-\frac{1}{2}} - \frac{r}{r'^2} \cos V,$$

in which r and r' are the radii vectores of the planets, and V the angle between these radii vectores.

The second term of this expression admits, after the development is effected, of being merged in the first by a simple and well-known modification of certain terms of the first; we shall therefore confine our attention to the first term. The following notation is used:

v, v', the true angular distances of the planets from their common node.
λ, λ', the mean values of v and v'.
y, the mutual inclination of the orbits.
σ, sine $\frac{1}{2}y$.
ρ, ρ', the logarithms of the radii vectores.
ν, ν', the logarithms of the mean distances.
e, e', the eccentricities.
g, g', the mean anomalies.
α, the ratio of the mean distances.

By substituting for $\cos V$ its known value,

$$\cos V = \cos v \cos v' + \sin v \sin v' \cos y$$

or

$$\cos V = (1 - \sigma^2) \cos (v - v') + \sigma^2 \cos (v + v'),$$

and then developing in cosines of multiples of v and v', we shall obtain the terms with which we are to set out. In order not to weary the reader with

what is not essential to the present object, we shall state only those conclusions which form the basis of the new method.

If we suppose the eccentricities to vanish, the value of R can be developed in the form

$$\begin{aligned} R = & \tfrac{1}{2}\Sigma A_i \cos(i\lambda' - i\lambda) \\ & + \quad \Sigma B_i \cos((i+1)\lambda' - (i-1)\lambda) \\ & + \quad \Sigma C_i \cos((i+2)\lambda' - (i-2)\lambda) \\ & + \text{etc.} \end{aligned} \qquad (a)$$

where the index i takes all integral values from positive to negative infinity, and A_i, B_i, C_i, etc. are functions of the mean distances and inclinations which admit of explicit development in powers of σ.

This expression for R may be thrown into the form

$$R = \Sigma_\nu \Sigma_\mu A_{\nu,\mu} \cos(\nu\lambda' + \mu\lambda) \qquad (1)$$

where μ and ν each assume all values from $-\infty$ to $+\infty$, but not independently, being subject to the single restriction that both values must be even, or both odd.

The coefficients A are homogeneous and of the degree -1 in a and a', admitting of being expressed indifferently in either of the forms

$$\frac{1}{a'}\phi\left(\frac{a}{a'}\right) \qquad \text{or} \qquad \frac{1}{a}\phi\left(\frac{a'}{a}\right).$$

It is in practice more convenient to choose the form in which the fraction $\frac{a}{a'}$ or $\frac{a'}{a}$ shall be less than unity, but, for the purposes of the present investigation, the choice is indifferent.

The original expression for R is of the form

$$R = f(v, v', r, r', \sigma); \qquad (2)$$

and, assuming the eccentricities of both planets to vanish, we have supposed it developed in the form

$$R_0 = f(\lambda, \lambda', a, a', \sigma). \qquad (3)$$

In order to express R as a function of the eccentricities and other elements of the two planets, we must now substitute v, v', r, and r' for λ, λ', a and a' in (3), or, which is the same thing, in (1), the quantities to be substituted being expressed in terms of the eccentricities and mean anomalies.

To continue the process we shall consider R as a function of the logarithms of the radii vectores of the planets, instead of the radii vectores themselves, putting

$$\begin{aligned} \rho &= \log r, & \rho' &= \log r', \\ \mathfrak{v} &= \log a, & \mathfrak{v}' &= \log a', \end{aligned} \qquad (4)$$

which gives $$\frac{da}{d\mathfrak{v}} = a\,; \qquad \frac{dr}{d\rho} = r.$$

It is not necessary to have either A or its derivatives expressed explicitly as a function of $\mathfrak{v}$ and $\mathfrak{v}'$, since we have, with respect to any function ϕ of a and a',

$$\frac{d\phi}{d\mathfrak{v}} = a\,\frac{d\phi}{da}\,; \qquad \frac{d\phi}{d\mathfrak{v}'} = a'\,\frac{d\phi}{da'}\,,$$

a form by which all the derivatives with respect to $\mathfrak{v}$ and $\mathfrak{v}'$ may be expressed as functions of a and a'.

All the coefficients of $A_{\mu,\nu}$ being homogeneous and of the degree -1 in a and a', we have

$$\frac{dA}{d\mathfrak{v}} + \frac{dA}{d\mathfrak{v}'} + A = 0.$$

We shall hereafter use a symbolic notation, putting D for the operation $\frac{d}{d\mathfrak{v}}$ and D' for $\frac{d}{d\mathfrak{v}'}$. All the derivatives of A with respect to $\mathfrak{v}$ and $\mathfrak{v}'$ being, like A itself, homogeneous and of the degree -1, we may put, in general,

$$D + D' = -1, \tag{5}$$

and may combine D and D' as if they were multipliers according to the usual rules for such symbols.

To effect the development we require we must, in (2), put

$$\begin{aligned} V &= \lambda + \phi\,(e\,, g), \\ \rho &= \mathfrak{v} + \psi\,(e\,, g), \\ V' &= \lambda' + \phi\,(e', g'), \\ \rho' &= \mathfrak{v}' + \psi\,(e', g')\,; \end{aligned} \tag{6}$$

the function ϕ representing the equation of the centre, and ψ the portion of $\log r$ which depends on the eccentricity. From these equations we see that considering R first as a function of V, V', ρ, and ρ', and then as a function of $\lambda, \lambda', \mathfrak{v}, \mathfrak{v}', e, e', g$ and g', we shall have for any compound derivative with respect to V and ρ

$$\frac{d^{m+m'+n+n'}R}{dV^m dV'^{m'} d\rho^n d\rho'^{n'}} = \frac{d^{m+m'+n+n'}R}{d\lambda^m d\lambda'^{m'} d\mathfrak{v}^n d\mathfrak{v}'^{n'}}\,,$$

that is, *any derivative with respect to* V, V', ρ, *or* ρ' *is found by taking the corresponding derivative of the developed function with respect to* $\lambda, \lambda', \mathfrak{v}$, *and* $\mathfrak{v}'$

Now, supposing that, in (2), V and ρ are replaced by their values in (6), we shall have

$$\begin{aligned}\frac{dR}{de} &= \frac{dR}{dV}\frac{dV}{de} + \frac{dR}{d\rho}\frac{d\rho}{de} \\ &= \frac{dV}{de}\frac{dR}{d\lambda} + \frac{d\rho}{de}\frac{dR}{d\nu}. \end{aligned} \tag{7}$$

This equation is the fundamental one in our method of development. By it we express the derivative of R with respect to e in terms of its derivatives with respect to λ and ν. Let us now differentiate this expression with respect to e n times in succession, representing by D_λ the operation $\frac{d}{d\lambda}$. We thus obtain

$$\begin{aligned}\frac{d^{n+1}R}{de^{n+1}} = D_\lambda &\left\{\frac{dV}{de}\frac{d^nR}{de^n} + \binom{n}{1}\frac{d^2V}{de^2}\frac{d^{n-1}R}{de^{n-1}} + \binom{n}{2}\frac{d^3V}{de^3}\frac{d^{n-2}R}{de^{n-2}} + \dots\right\} \\ + D_\nu &\left\{\frac{d\rho}{de}\frac{d^nR}{de^n} + \binom{n}{1}\frac{d^2\rho}{de^2}\frac{d^{n-1}R}{de^{n-1}} + \binom{n}{2}\frac{d^3\rho}{de^3}\frac{d^{n-2}R}{de^{n-2}} + \dots\right\}\end{aligned} \tag{8}$$

Thus, we have expressed the derivative of any order with respect to e in terms of the derivatives of lower orders, and, by successive substitutions, this derivative will be expressed in terms of derivatives of R with respect to λ and ν.

The coefficient of e^{n+1} in R is found by putting $e = 0$ in (8), and dividing by $1.2.3\dots n+1$. In strictness we should suppose $e = 0$ only after differentiating with respect to λ, but it is evident that the two operations may be interchanged without affecting the result. If we represent this coefficient by $R^{(n+1)}$, and replace the derivatives with respect to e by the corresponding values of this coefficient, namely,

$$\begin{aligned}\frac{dR_0}{de} &= R^{(1)} \\ \frac{d^2R_0}{de^2} &= 2!\,R^{(2)} \\ \frac{d^3R_0}{de^3} &= 3!\,R^{(3)} \\ &\;\;\vdots \\ \frac{d^nR_0}{de^n} &= n!\,R^{(n)} \\ \text{etc.}\quad & \quad \text{etc.}\end{aligned}$$

If, also, we represent, as in these last equations, by the subscript zero the operation of putting $e = 0$ after differentiation, we shall have

$$R^{(n+1)} = D_\lambda \left\{ \frac{n\,!}{(n+1)\,!} \frac{d\,v_0}{de} R^{(n)} + \binom{n}{1} \frac{(n-1)\,!}{(n+1)\,!} \frac{d^2 v_0}{de^2} R^{(n-1)} + \text{etc.} \ldots . \right\}$$
$$+ D_\nu \left\{ \frac{n\,!}{(n+1)\,!} \frac{d\rho_0}{de} R^{(n)} + \binom{n}{1} \frac{(n-1)\,!}{(n+1)\,!} \frac{d^2 \rho_0}{de^2} R^{(n-1)} + \text{etc.} \ldots . \right\}.$$

If we represent by v_n and ρ_n the coefficients of e^n in v and ρ respectively, we shall have

$$\frac{d^n v_0}{de^n} = n\,!\, v_n; \qquad \frac{d^n \rho_0}{de^n} = n\,!\, \rho_n.$$

Substituting these values in the above equations, we find by simple reductions

$$(n+1) R^{n+1} = D_\lambda \{ v_1 R^{(n)} + 2\, v_2 R^{(n-1)} + 3\, v_3 R^{(n-2)} + \ldots . + (n+1) v_{n+1} R_0 \}$$
$$+ D_\nu \{ \rho_1 R^{(n)} + 2\, \rho_2 R^{(n-1)} + 3\, \rho_3 R^{(n-2)} + \ldots . + (n+1) \rho_{n+1} R_0 \}. \quad (9)$$

Thus, each coefficient is expressed as a linear function of the derivatives of the coefficients of lower orders. We have next to substitute for v_i and ρ_i their values in terms of g. We have in general

$$i v_i = \tfrac{1}{2} \sum_{j=-i}^{j=i} k_j^{(i)} \sin jg; \qquad i \rho_i = \tfrac{1}{2} \sum_{j=-i}^{j=i} h_j^{(i)} \cos jg \qquad (10)$$

from the developments of the elliptic motion, but it is to be remarked that j does not assume all values between the limits $+\,i$ and $-\,i$, but only every alternate value. The special values of k and h to terms of the seventh order are shown in the following scheme. The values for negative values of j are formed from those for positive values by the formulæ

$$k_{-j}^{(i)} = -\, k_j^{(i)}; \qquad h_{-j}^{(i)} = h_j^{(i)}.$$

In strictness it is not necessary to suppose j negative at all, since the complete value of $4\,\rho_4$, for example, is

$$4\,\rho_4 = \tfrac{1}{2}\, h_0^{\text{IV}} + h_2^{\text{IV}} \cos 2\,g + h_4^{\text{IV}} \cos 4\,g,$$

but it will be found convenient in forming the required functions to suppose j negative in some cases.

Values of h *and* k.

k'_1 $+2$	h'_1 -1	k'_{-1} -2	h'_{-1} -1	0	0	0	0
k''_2 $+\frac{5}{2}$	h''_2 $-\frac{3}{2}$	k''_0 0	h''_0 $+1$	k''_{-2} $-\frac{5}{2}$	h''_{-2} $-\frac{3}{2}$	0	0
k'''_3 $+\frac{13}{4}$	h'''_3 $-\frac{17}{8}$	k'''_1 $-\frac{3}{4}$	h'''_1 $+\frac{9}{8}$	k'''_{-1} $+\frac{3}{4}$	h'''_{-1} $+\frac{9}{8}$	k'''_{-3} $-\frac{13}{4}$	h'''_{-3} $-\frac{17}{8}$
k^{IV}_4 $+\frac{103}{24}$	h^{IV}_4 $-\frac{71}{24}$	k^{IV}_2 $-\frac{11}{6}$	h^{IV}_2 $+\frac{11}{6}$	k^{IV}_0 0	h^{IV}_0 $+\frac{1}{4}$	k^{IV}_{-2} $+\frac{11}{6}$	h^{IV}_{-2} $+\frac{11}{6}$
k^{V}_5 $+\frac{1097}{192}$	h^{V}_5 $-\frac{523}{128}$	k^{V}_3 $-\frac{215}{64}$	h^{V}_3 $+\frac{385}{128}$	k^{V}_1 $+\frac{25}{96}$	h^{V}_1 $+\frac{5}{64}$	k^{V}_{-1} $-\frac{25}{96}$	h^{V}_{-1} $+\frac{5}{64}$
k^{VI}_6 $+\frac{1223}{160}$	h^{VI}_6 $-\frac{899}{160}$	k^{VI}_4 $-\frac{451}{80}$	h^{VI}_4 $+\frac{387}{80}$	k^{VI}_2 $+\frac{17}{32}$	h^{VI}_2 $-\frac{9}{32}$	k^{VI}_0 0	h^{VI}_0 $+\frac{1}{8}$
k^{VII}_7 $+\frac{330911}{32256}$	h^{VII}_7 $-\frac{355081}{46080}$	k^{VII}_5 $-\frac{41699}{4608}$	h^{VII}_5 $+\frac{70273}{9216}$	k^{VII}_3 $+\frac{665}{512}$	h^{VII}_3 $-\frac{5201}{5120}$	k^{VII}_1 $+\frac{749}{4608}$	h^{VII}_1 $+\frac{889}{9216}$

To commence the development, let us take any one term of (1), and for brevity put

$$N = \nu\lambda' + \mu\lambda,$$

and omit writing the indices μ and ν. The term will then be

$$R_0 = R^{(0)} = A \cos N,$$

and its derivatives with respect to λ and ϖ will be

$$\frac{dR^{(0)}}{d\lambda} = -\mu A \sin N,$$

$$\frac{dR^{(0)}}{d\varpi} = DA \cos N.$$

The general formula (9) gives, by putting $n = 0$,

$$R^{(1)} = \nu_1 \frac{dR^{(0)}}{d\lambda} + \rho_1 \frac{dR^{(0)}}{d\upsilon},$$

$$= -\mu k_1' A \sin g \sin N + h_1' D A \cos g \cos N,$$

whence

$$2R^{(1)} = (\mu k_1' A \quad + h_1' DA) \cos (N + g)$$
$$+ (-\mu k_1' A + h_1' DA) \cos (N - g).$$

By the repeated application of (9), supposing n successively equal to 1, 2, 3, etc., we shall obtain the successive values of $R^{(n)}$. This process will be facilitated by finding a general formula for passing from $R^{(n)}$ to $R^{(n+1)}$. For this purpose let us put

$$P_j^n, \text{ the coefficient of } \cos (N + jg) \text{ in } R^{(n)},$$

we then have, in the special cases $n = 0$ and $n = 1$,

$$P_0^0 = A,$$
$$P_1^1 = (\mu k_1' \quad + h_1' D) A,$$
$$P_{-1}^1 = (-\mu k_1' + h_1' D) A,$$

and in the general case

$$R^{(n)} = P_n^n \cos (N + ng) + P_{n-2}^n \cos (N + (n-2)\, g) + \cdots + P_{-n}^n \cos (N - ng),$$

the index j taking each alternate value from $+n$ to $-n$. Differentiating with respect to λ and υ, we shall have

$$D_\lambda R^{(n)} = -\mu P_n^n \sin (N + ng) - \mu P_{n-2}^n \sin (N + (n-2)\, g) - \text{etc.}$$

$$D_\upsilon R_{(n)} = D P_n^n \cos (N + ng) + D P_{n-2}^n \cos (N + (n-2)\, g) + \text{etc.}$$

Putting side by side, and writing in the most condensed form, the pairs of factors which enter into (9), we find them as follows:

$$D_\lambda R^{(n)} = -\mu \sum_{j=-n}^{j=+n} P_j^n \sin (N + jg) \qquad \nu_1 = k_1' \sin g$$

$$D_\lambda R^{(n-1)} = -\mu \sum_{j=-n+1}^{j=n-1} P_j^{n-1} \sin (N + jg) \qquad 2\nu_2 = k_2'' \sin 2g$$

$$D_\lambda R^{(n-2)} = -\mu \sum_{j=-n+2}^{j=n-2} P_j^{n-2} \sin (N + jg) \qquad 3\nu_3 = k_3''' \sin 3g + k_1''' \sin g$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots$$

$$D_\lambda R^{(0)} = -\mu P_0^0 \sin N \qquad (n+1)\, \nu_{n+1} = \tfrac{1}{2} \sum_{i=-n-1}^{i=n+1} k_i^{(n+1)} \sin ig$$

$$
\begin{aligned}
D\nu R^{(n)} &= D\sum_{j=-n}^{j=n} P_j^n \cos(N+jg) & \rho_1 &= h'_1 \cos g \\
D\nu R^{(n-1)} &= D\sum_{j=-n+1}^{j=n-1} P_j^{n-1} \cos(N+jg) & 2\,\rho_2 &= h''_2 \cos 2\,g + \tfrac{1}{2}\,h''_0 \\
D\nu R^{(n-2)} &= D\sum_{j=-n+2}^{j=n-2} P_j^{n-2} \cos(N+jg) & 3\,\rho_3 &= h'''_3 \cos 3\,g + h'''_1 \cos g \\
&\vdots & &\vdots \\
D\nu R^{(0)} &= DP_0^0 \cos N & (n+1)\,\rho_{n+1} &= \tfrac{1}{2}\sum_{i=-n-1}^{i=n+1} h_i^{(n+1)} \cos ig.
\end{aligned}
$$

If we form the products of these quantities according to the formula (9), and compare the coefficients of the several angles, we find

$$
\begin{aligned}
2\,(n+1)\,P^{n+1}_{n+1} = {} & (\mu k'_1 + h'_1 D)\,P^n_n \\
& + (\mu k''_2 + h''_2 D)\,P^{n-1}_{n-1} \\
& + (\mu k'''_3 + h'''_3 D)\,P^{n-2}_{n-2}\,. \\
& + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
& + (\mu k^{(n+1)}_{n+1} + h^{(n+1)}_{n+1} D)\,P^0_0
\end{aligned}
$$

$$
\begin{aligned}
2\,(n+1)\,P^{n+1}_{n-1} = {} & (\mu k'_1 + h'_1 D)\,P^n_{n-2} & & + (\mu k'_{-1} + h'_{-1} D)\,P^n_n \\
& + (\mu k''_2 + h''_2 D)\,P^{n-1}_{n-3} & & + \qquad\qquad h''_0 D P^{n-1}_{n-1} \\
& + \text{etc.} & & + (\mu k'''_1 + h'''_1 D)\,P^{n-2}_{n-2} \qquad\qquad (11) \\
& + (\mu k^{(n)}_n + h^{(n)}_n D)\,P^1_{-1} & & + \text{etc} \\
& & & + (\mu k^{(n+1)}_{n-1} + h^{(n+1)}_{n-1} D)\,P^0_0
\end{aligned}
$$

$$
\begin{aligned}
2(n+1)\,P^{n+1}_{n-3} = {} & (\mu k'_1 \;\; + h'_1 D)\,P^n_{n-4} + (\mu k'_{-1} + h'_{-1} D)\,P^n_{n-2} \\
& + (\mu k''_2 \;\; + h''_2 D)\,P^{n-1}_{n-5} + \qquad\qquad h''_0 D P^{n-1}_{n-3} + (\mu k''_{-2} \; + h''_{-2} D)\,P^{n-1}_{n-1} \\
& + (\mu k'''_3 \;\; + h'''_3 D)\,P^{n-2}_{n-6} + (\mu k'''_1 \;\; + h'''_1 D)\,P^{n-2}_{n-4} + (\mu k'''_{-1} \; + h'''_{-1} D)\,P^{n-2}_{n-2} \\
& + \qquad\qquad\qquad\qquad + \qquad\qquad\qquad\qquad + \\
& \vdots \qquad\qquad\qquad\qquad\quad \vdots \qquad\qquad\qquad\qquad\quad \vdots \\
& + (\mu k^{(n-1)}_{n-1} + h^{(n-1)}_{n-1})\,P^2_{-2} + (\mu k^{(n)}_{n-2} + h^{(n)}_{n-2} D)\,P'_{-1} + (\mu k^{(n+1)}_{n-3} + h^{(n+1)}_{n-3} D)\,P^0_0
\end{aligned}
$$

To show the law of progression of the several classes of terms, I have purposely written some of the coefficients k and h with negative indices instead of using the corresponding positive ones, which, in the actual development, will be

substituted for them. The law can be seen by induction and comparison without a full statement of it. As we diminish the value of the lower exponent of P by successive steps, two units at a time, the number of columns of products increases by one at each step, but the number of products in each column diminishes in consequence of one of the factors vanishing whenever a lower exponent in h, k, or P exceeds the upper one in absolute value. Supposing the successive values of P_j^{n+1} to be written in this form with continually diminishing values of j until we reach the value $j = -(n+1)$, to this value would correspond $n+2$ columns, of which, however, the first would entirely vanish, as would every term of the remaining ones, except the first. Placing these terms in a column, they will be as follows:—

$$
\begin{array}{ll}
P_{-n-1}^{n+1} = (\mu k'_{-1} + h'_{-1}D)\, P_{-n}^{n} & = (-\mu k'_1 + h'_1 D)\, P_{-n}^{n} \\
\qquad + (\mu k''_{-2} + h''_{-2}D)\, P_{-n+1}^{n-1} & + (-\mu k''_2 + h''_2 D)\, P_{-n+1}^{n-1} \\
\qquad + (\mu k'''_{-3} + h'''_{-3}D)\, P_{-n+2}^{n-2} & + (-\mu k'''_3 + h'''_3 D)\, P_{-n+2}^{n-2} \\
\qquad + & + \\
\qquad \cdot & \cdot \\
\qquad \cdot & \cdot \\
\qquad \cdot & \cdot \\
\qquad + (\mu k_{-n-1}^{(n+1)} + h_{-n-1}^{(n+1)}\, D)\, P_0^0 & + (-\mu k_{n+1}^{(n+1)} + h_{n+1}^{(n+1)} D)\, P_0^0
\end{array}
$$

This is a particular case of the general law of formation of the values of P when j is negative, which law may be expressed as follows:—

Each value of P *with* j *negative may be formed from the corresponding value for* j *positive by changing the sign of* μ, *and of the lower exponents in all the values of* P *which enter into it.*

We have next to consider the development with respect to the powers of e'. What we have hitherto done has been to take

$$R = f(v, \rho, v', \rho')$$

and substituting

$$v = \lambda + \phi, (e, g);$$

$$\rho = \varpi + \psi, (e, g),$$

to find the coefficients $R^{(n)}$ of the development with respect to e only. We have written λ' and ϖ' instead of v and ρ in giving the values of the coefficients. Now, when we replace λ' and ϖ' by v' and ρ', expressed in terms of the elements, each coefficient $R^{(n)}$ will become a function of v' and ρ', and hence of e', g', etc. We shall now proceed to develop each of these coefficients in powers of e' by the same process which was followed in developing R in powers of e. Let

$$P_j^n \cos(\nu\lambda' + \mu\lambda + jg)$$

be any term of $R^{(n)}$, and, having replaced λ' by v' and ϖ' by ρ' in this term, let us suppose

$$v' = \lambda' + \phi(e', g'),$$
$$\rho' = \varpi' + \psi(e', g');$$

and let us put

$R^{n,n'}$, the coefficient of $e'^{n'}$ in this development.

We shall then have, as in (9),

$$(n'+1)\,R^{n,n'+1} = v'_1 D_{\lambda'} R^{n,n'} + 2\,v'_2 D_{\lambda'} R^{n,n'-1} + \ldots + (n'+1)\,v'_{n'+1} R^{n,0}$$
$$+ \rho'_1 D' R^{n,n'} + 2\,\rho'_2 D' R^{n,n'-1} + \ldots + (n'+1)\,\rho'_{n'+1} R^{n,0}. \quad (12)$$

The expressions for v' and ρ' will be the same as those for v and ρ in (10), except that g' is to be substituted for g. We may now investigate the general law of development in the same way as before. Putting, for brevity,

$$N' = \mu\lambda + \nu\lambda' + jg = N + jg,$$

so that

$$P^n_j \cos N'$$

is any term of $R^{(n)}$ or, which is the same thing, of $R^{n,0}$, let us represent the corresponding terms in $R^{n,n'}$ by

$$R^{n,n'} = P^{n,n'}_{j,n'} \cos(N' + n'g') + P^{n,n'}_{j,n'-2} \cos(N' + (n'-2)g') + \ldots + P^{n,n'}_{j,-n'} \cos(N' - n'g').$$

Then, proceeding as before, we shall find

$$2(n'+1)\,P^{n,n'+1}_{j,-n'+1} = (\nu k'_1 + h'_1 D')\,P^{n,n'}_{j,n'}$$
$$+ (\nu k''_2 + h''_2 D')\,P^{n,n'-1}_{j,n'-1}$$
$$+ \ldots \ldots \ldots$$
$$+ (\nu k^{(n'+1)}_{n'+1} + h^{(n'+1)}_{n'+1} D')\,P^{n,0}_{j,0}. \quad (13)$$

All the other coefficients can be formed from the corresponding ones of (11) by writing

ν for μ,

D' for D,

P^n_j for P,

n' for n.

We have now an important law of this development of R to bring out. First, in the equations (11), by supposing in succession $n = 1$, $n = 2$, $n = 3$, etc., and by continually substituting in each set of terms the values of those of a lower order, we shall finally express all the values of P^n_j in terms of $P^0_0 = A$ and its successive

derivatives with respect to v. Moreover, the operation of forming these derivatives being always linear, we can combine all the operations represented by the symbols $\mu k + hD$ as if D represented a coefficient; that is, having the quantity

$$(\mu k + hD)\,(\mu k' + h'D)\,\ldots .\,(\mu k_n + h_n D)\,P_0^0$$

we can multiply these several symbols as if D were a coefficient. By this operation, putting A for P_0^0, we shall finally obtain an expression of the form

$$P_j^n = \Pi_j^n A,$$

in which Π_j^n represents an entire function of μ and D of the degree n.

Secondly, by treating the equations represented by (13) in the same manner, we shall be able to represent each value of $P_{j\,;\,j'}^{n\,;\,n'}$ in the form

$$P_{j\,;\,j'}^{n\,;\,n'} = \Pi_{j'}^{n'}\,(P_{j\,;\,0}^{n\,;\,0} = P_j^n)$$

in which $\Pi_{j'}^{n'}$ represents an entire function of ν and D' of the degree n'. Substituting for P_j^n its value just given, we shall have

$$P_{j\,;\,j'}^{n\,;\,n'} = \Pi_{j'}^{n'}\Pi_j^n A \tag{14}$$

in which the two symbols can be combined by the rule of multiplication. *It thus appears that when we have found the development in powers of* e *for* $\mathrm{e}' = 0$, *and that in* e′ *for* $\mathrm{e} = 0$, *we have solved the whole problem, and the terms multiplied by any product of a power of* e *by a power of* e′ *can then be found by a symbolic multiplication.*

It may not be amiss to recapitulate the result which we have reached. Suppose that we develop R in powers of e on the supposition $e' = 0$, and that any term of this development is represented in the form

$$R = e^n \Pi_j^n A \cos(N + jg),$$

A being a function of the mean distances, Π_j^n, an operating symbol, and N a function of λ and λ' which does not contain g. Suppose, next, that we develop R in powers of e', putting $e = 0$, and that any term of this development is represented by

$$R = e'^{n'} \Pi_{j'}^{n'} A \cos(N + j'g'),$$

then the coefficient of $\cos(N + jg + j'g')$ in the complete development will be represented by

$$e^n e'^{n'} \Pi_j^n \Pi_{j'}^{n'} A. \tag{15}$$

To proceed to the actual development, it is necessary to form the symbolic factors represented by Π_j^n and $\Pi_{j'}^{n'}$. This we may do from the forms (11) by substituting for h and k their numerical values, and substituting the symbol

Π_j^i for P_j^i, remembering that $\Pi_0^0 = 1$. We thus have, for each successive value of n,

$$\begin{aligned}
2(n+1)\,\Pi_{n+1}^{n+1} = {} & (2\mu - D)\,\Pi_n^n \\
& + \left(\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi_{n-1}^{n-1} \\
& + \left(\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi_{n-2}^{n-2} \\
& + \left(\frac{103}{24}\mu - \frac{71}{24}D\right)\Pi_{n-3}^{n-3} \\
& + \left(\frac{1097}{192}\mu - \frac{523}{128}D\right)\Pi_{n-4}^{n-4} \\
& + \left(\frac{1223}{160}\mu - \frac{899}{160}D\right)\Pi_{n-5}^{n-5} \\
& + \text{etc.}
\end{aligned}$$

the series terminating with Π_0^0.

$$\begin{aligned}
2(n+1)\,\Pi_{n-1}^{n+1} = {} & (2\mu - D)\,\Pi_{n-2}^{n} && + (-2\mu - D)\,\Pi_n^n \\
& + \left(\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi_{n-3}^{n-1} && + D\,\Pi_{n-1}^{n-1} \\
& + \left(\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi_{n-4}^{n-2} && + \left(-\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi_{n-2}^{n-2} \\
& + \text{etc.} && + \text{etc.} \\
& + (k_n^{(n)}\mu + h_n^{(n)}D)\,\Pi'_{-1} && + (k_{n-1}^{(n+1)}\mu + h_{n-1}^{(n+1)}D)\,\Pi_0^0
\end{aligned}$$

$$\begin{aligned}
2(n+1)\,\Pi_{n-3}^{n+1} = {} & (2\mu - D)\,\Pi_{n-4}^{n} && + (-2\mu - D)\,\Pi_{n-2}^{n} \\
& + \left(\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi_{n-5}^{n-1} && + D\,\Pi_{n-3}^{n-1} + \left(-\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi_{n-1}^{n-1} \\
& + \left(\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi_{n-6}^{n-2} && + \left(-\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi_{n-4}^{n-2} + \left(\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi_{n-2}^{n-2} \\
& + \text{etc.} && + \text{etc.} \qquad\qquad + \text{etc.} \\
& + (k_{n-1}^{(n-1)}\mu + h_{n-1}^{(n-1)}D)\,\Pi_{-2}^{2} && + (k_{n-2}^{(n)}\mu + h_{n-2}^{(n)}D)\,\Pi_{-1}^{1} + (k_{n-3}^{(n+1)}\mu + h_{n-3}^{(n+1)}D)\,\Pi_0^0
\end{aligned}$$

$$\begin{aligned}
2(n+1)\,\Pi_{n-5}^{n+1} = {} & (2\mu - D)\,\Pi_{n-6}^{n} && + (-2\mu - D)\,\Pi_{n-4}^{n} \\
& + \left(\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi_{n-7}^{n-1} && + D\,\Pi_{n-5}^{n-1} + \left(-\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi_{n-3}^{n-1} \\
& + \left(\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi_{n-8}^{n-2} && + \left(-\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi_{n-6}^{n-2} + \left(\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi_{n-4}^{n-2} \\
& && \qquad + \left(-\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi_{n-2}^{n-2} \\
& + \text{etc.} && + \text{etc.} \qquad\qquad + \text{etc.} \\
& + (k_{n-2}^{(n-2)}\mu + h_{n-2}^{(n-2)}D)\,\Pi_{-3}^{3} && + (k_{n-3}^{(n-1)}\mu + h_{n-3}^{(n-1)}D)\,\Pi_{-2}^{2} + (k_{n-4}^{(n)}\mu + h_{n-4}^{(n)}D)\,\Pi_{-1}^{1} \\
& && \qquad + (k_{n-5}^{(n+1)}\mu + h_{n-5}^{(n+1)}D)\,\Pi_0^0
\end{aligned}$$

$$2(n+1)\,\Pi^{n+1}_{-n-1} = (-2\mu - D)\,\Pi^{n}_{-n}$$
$$+\left(-\tfrac{5}{2}\mu - \tfrac{3}{2}D\right)\Pi^{n-1}_{-n+1}$$
$$+\left(-\tfrac{13}{4}\mu - \tfrac{17}{8}D\right)\Pi^{n-2}_{-n+2}$$
$$+\text{ etc.}$$
$$+\left(-k^{(n+1)}_{n+1}\mu + h^{(n+1)}_{n+1}D\right)\Pi^{0}_{0}$$

$$\begin{aligned}2(n+1)\,\Pi^{n+1}_{-n+1} = {} & (-2\mu - D)\,\Pi^{n}_{-n+2} && +(2\mu - D)\,\Pi^{n}_{-n}\\
& +\left(-\tfrac{5}{2}\mu - \tfrac{3}{2}D\right)\Pi^{n-1}_{-n+3} && +D\Pi^{n-1}_{-n+1}\\
& +\left(-\tfrac{13}{4}\mu - \tfrac{17}{8}D\right)\Pi^{n-2}_{-n+4} && +\left(\tfrac{3}{4}\mu + \tfrac{9}{8}D\right)\Pi^{n-2}_{-n+2}\\
& +\text{ etc.} && +\text{ etc.}\\
& +\left(-k^{(n)}_{n}\mu + h^{(n)}_{n}D\right)\Pi^{1}_{1} && +\left(-k^{(n+1)}_{n-1}\mu + h^{(n+1)}_{n-1}D\right)\Pi^{0}_{0}\end{aligned}$$

The actual computation of Π^{n}_{j} for negative values of j is not necessary, since its values may be obtained from those for positive j by changing the sign of μ. These numerical coefficients may be continued to any extent by means of the scheme of values of h and k already given, it being remarked that the successive columns appear in the same order as in the scheme.

We are now ready to proceed with the actual computation of the functions $P^{n;\,n'}_{j;\,j'}$, or, which is the same thing, of the symbolic functions $\Pi^{n;\,n'}_{j;\,j'}$ which express the values of $P^{n;\,n'}_{j;\,j'}$ when considered as operators on the functions A_i, B_i, etc. We shall give only a few of these functions for the purpose of illustrating the method.

We consider, firstly, the general values of $\Pi^{n;\,0}_{j;\,0}$ and of $\Pi^{0;\,n'}_{0;\,j'}$ which arise when, in the general term of R for circular orbits,

$$R = A\cos(\nu\lambda' + \mu\lambda),$$

we substitute for the mean values of the radii vectores and longitudes those which correspond to the elliptic motion. The functions of Π for values of j and j' to the fourth order, inclusive, are then found to be as follows. It is not necessary to write the functions for negative values of j or j', because they are found from the corresponding ones for positive values by simply changing the sign of μ and D, but a few are given for perspicuity.

$$\Pi^0_0 = 1$$

$$\Pi^1_1 = \mu - \frac{1}{2} D$$

$$\Pi^1_{-1} = -\mu - \frac{1}{2} D$$

$$\Pi^2_2 = \frac{1}{2}\mu^2 + \frac{5}{8}\mu + \left(-\frac{1}{2}\mu - \frac{3}{8}\right) D + \frac{1}{8} D^2$$

$$\Pi^2_0 = -\mu^2 \qquad\qquad + \frac{1}{4} D + \frac{1}{4} D^2$$

$$\Pi^2_{-2} = \frac{1}{2}\mu^2 - \frac{5}{8}\mu + \left(\frac{1}{2}\mu - \frac{3}{8}\right) D + \frac{1}{8} D^2$$

$$\Pi^3_3 = \frac{1}{6}\mu^3 + \frac{5}{8}\mu^2 + \frac{13}{24}\mu + \left(-\frac{1}{4}\mu^2 - \frac{11}{16}\mu - \frac{17}{48}\right) D + \left(\frac{1}{8}\mu + \frac{3}{16}\right) D^2 - \frac{1}{48} D^3$$

$$\Pi^3_1 = -\frac{1}{2}\mu^3 - \frac{5}{8}\mu^2 - \frac{1}{8}\mu + \left(\frac{1}{4}\mu^2 + \frac{5}{16}\mu + \frac{3}{16}\right) D + \left(\frac{1}{8}\mu + \frac{1}{16}\right) D^2 - \frac{1}{16} D^3.$$

$$\Pi^3_{-1} = \frac{1}{2}\mu^3 - \frac{5}{8}\mu^2 + \frac{1}{8}\mu + \left(\frac{1}{4}\mu^2 - \frac{5}{16}\mu + \frac{3}{16}\right) D + \left(-\frac{1}{8}\mu + \frac{1}{16}\right) D^2 - \frac{1}{16} D^3.$$

$$\Pi^3_{-3} = -\frac{1}{6}\mu^3 + \frac{5}{8}\mu^2 - \frac{13}{24}\mu + \left(-\frac{1}{4}\mu^2 + \frac{11}{16}\mu - \frac{17}{48}\right) D + \left(-\frac{1}{8}\mu + \frac{3}{16}\right) D^2 - \frac{1}{48} D^3$$

$$\Pi^4_4 = \frac{1}{24}\mu^4 + \frac{5}{16}\mu^3 + \frac{283}{384}\mu^2 + \frac{103}{192}\mu + \left(-\frac{1}{12}\mu^3 - \frac{1}{2}\mu^2 - \frac{5[\text{illegible}]}{64}\mu - \frac{71}{192}\right) D$$
$$+ \left(\frac{1}{16}\mu^2 + \frac{17}{64}\mu + \frac{95}{384}\right) D^2 + \left(-\frac{1}{48}\mu - \frac{3}{64}\right) D^3 + \frac{1}{384} D^4$$

$$\Pi^4_2 = -\frac{1}{6}\mu^4 - \frac{5}{8}\mu^3 - \frac{2}{3}\mu^2 - \frac{11}{48}\mu + \left(\frac{1}{6}\mu^3 + \frac{1}{2}\mu^2 + \frac{47}{96}\mu + \frac{11}{48}\right) D$$
$$+ \left(\frac{1}{32}\mu - \frac{1}{96}\right) D^2 + \left(-\frac{1}{24}\mu - \frac{1}{16}\right) D^3 + \frac{1}{96} D^4$$

$$\Pi^4_0 = \frac{1}{4}\mu^4 - \frac{9}{64}\mu^2 + \frac{1}{32} D + \left(-\frac{1}{8}\mu^2 - \frac{1}{64}\right) D^2 - \frac{1}{32} D^3 + \frac{1}{64} D^4$$

The values of Π' being formed, as already shown, by simply changing μ into ν and D into D', it is unnecessary to write them as functions of D'. As it will be convenient to have but one form of derivative, they should be transformed, the symbol D' being replaced by D by writing

$$D' = -(1 + D).$$

We thus find

$$\Pi^{0\,;\,1}_{0\,;\,1} = \nu + \frac{1}{2} + \frac{1}{2} D$$

$$\Pi^{0\,;\,2}_{0\,;\,2} = \frac{1}{2}\nu^2 + \frac{9}{8}\nu + \frac{1}{2} + \left(\frac{1}{2}\nu + \frac{5}{8}\right) D + \frac{1}{8} D^2$$

$$\Pi^{0\,;\,2}_{0\,;\,0} = -\nu^2 \qquad\qquad + \frac{1}{4} D + \frac{1}{4} D^2$$

$$\Pi^{0\,;\,3}_{0\,;\,3} = \frac{1}{6}\nu^3 + \frac{7}{8}\nu^2 + \frac{65}{48}\nu + \frac{9}{16} + \left(\frac{1}{4}\nu^2 + \frac{15}{16}\nu + \frac{19}{24}\right) D + \left(\frac{1}{8}\nu + \frac{1}{4}\right) D^2 + \frac{1}{48} D^3$$

$$\Pi^{0\,;\,3}_{0\,;\,1} = -\frac{1}{2}\nu^3 - \frac{7}{8}\nu^2 - \frac{5}{16}\nu - \frac{1}{16} + \left(-\frac{1}{4}\nu^2 - \frac{1}{16}\nu + \frac{1}{8}\right) D + \left(\frac{1}{8}\nu + \frac{1}{4}\right) D^2 + \frac{1}{16} D^3$$

$$\Pi^{0\,;\,4}_{0\,;\,4} = \frac{1}{24}\nu^4 + \frac{19}{48}\nu^3 + \frac{499}{384}\nu^2 + \frac{323}{192}\nu + \frac{2}{3} + \left(\frac{1}{12}\nu^3 + \frac{5}{8}\nu^2 + \frac{93}{64}\nu + \frac{65}{64}\right) D$$
$$+ \left(\frac{1}{16}\nu^2 + \frac{21}{64}\nu + \frac{155}{384}\right) D^2 + \left(\frac{1}{48}\nu + \frac{11}{192}\right) D^3 + \frac{1}{384} D^4.$$

$$\Pi^{0\,;\,4}_{0\,;\,2} = -\frac{1}{6}\nu^4 - \frac{19}{24}\nu^3 - \frac{7}{6}\nu^2 - \frac{31}{48}\nu - \frac{1}{6} + \left(-\frac{1}{6}\nu^3 - \frac{1}{2}\nu^2 - \frac{29}{96}\nu - \frac{1}{48}\right) D$$
$$+ \left(\frac{5}{32}\nu + \frac{23}{96}\right) D^2 + \left(\frac{1}{24}\nu + \frac{5}{48}\right) D^3 + \frac{1}{96} D^4$$

$$\Pi^{0\,;\,4}_{0\,;\,0} = \frac{1}{4}\nu^4 - \frac{17}{64}\nu^2 + \left(-\frac{1}{4}\nu^2 + \frac{3}{32}\right) D + \left(-\frac{1}{8}\nu^2 + \frac{11}{64}\right) D^2 + \frac{3}{32} D^3 + \frac{1}{64} D^4.$$

The symbolic products

$$\Pi^{n\,;\,n'}_{j\,;\,j'} = \Pi^{n}_{j} \times \Pi^{0\,;\,n'}_{0\,;\,j'}$$

are functions of both the indices μ and ν. Instead of using them in their general form, it is more convenient to apply them to the separate terms of the development (a). To develop the first term, we put

$$\mu = -i\,;\ \nu = i.$$

For the second,

$$\mu = -i + 1\,;\ \nu = i + 1, \text{ etc.}$$

To show the forms to which we are thus led, the following exhibit of the first four orders of terms arising from the first term of (a) is presented. The symbol (i) indicates that in Π, μ is changed to $-i$ and ν to i.

Developed Value of R.

$$
\begin{aligned}
R = \quad & \tfrac{1}{2}\,\{1 + e^2\Pi^2_0 + e'^2\Pi^{0;2}_{0,0} + e^4\Pi^4_0 + e^2e'^2\Pi^{2;2}_{0,0} + e'^4\Pi^{0;4}_{0,0} + \text{etc.}\}^{(0)} A_0 \\
& + \sum_{i=1}^{i=\infty} \\
& \{1 + e^2\Pi^2_0 + e'^2\Pi^{0;2}_{0,0} + e^4\Pi^{4;0}_{0,0} + e^2e'^2\Pi^{2;2}_{0,0} + e'^4\Pi^{0;4}_{0,0} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda) \\
& + \sum_{i=-\infty}^{i=+\infty} \\
+ e \quad & \{\Pi^{1;0}_{1,0} + e^2\Pi^{3;0}_{1,0} + e'^2\Pi^{1;2}_{1,0} + e^4\Pi^{5;0}_{1,0} + e^2e'^2\Pi^{3;2}_{1,0} + e'^4\Pi^{1;4}_{1,0} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g) \\
+ e' \quad & \{\Pi^{0;1}_{0,1} + e^2\Pi^{2;1}_{0,1} + e'^2\Pi^{0;3}_{0,1} + e^4\Pi^{4;1}_{0,1} + e^2e'^2\Pi^{2;3}_{0,1} + e'^4\Pi^{0;5}_{0,1} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g') \\
\\
+ e^2 \quad & \{\Pi^{2;0}_{2,0} + e^2\Pi^{4;0}_{2,0} + e'^2\Pi^{2;2}_{2,0} + e^4\Pi^{6;0}_{2,0} + e^2e'^2\Pi^{4;2}_{2,0} + e'^4\Pi^{2;4}_{2,0} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 2g) \\
+ ee' \quad & \{\Pi^{1;1}_{1,1} + e^2\Pi^{3;1}_{1,1} + e'^2\Pi^{1;3}_{1,1} + e^4\Pi^{5;1}_{1,1} + e^2e'^2\Pi^{3;3}_{1,1} + e'^4\Pi^{1;5}_{1,1} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g' + g) \\
+ e'^2 \quad & \{\Pi^{0;2}_{0,2} + e^2\Pi^{2;2}_{0,2} + e'^2\Pi^{0;4}_{0,2} + e^4\Pi^{4;2}_{0,2} + e^2e'^2\Pi^{2;4}_{0,2} + e'^4\Pi^{0;6}_{0,2} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 2g') \\
\\
+ e^3 \quad & \{\Pi^{3;0}_{3,0} + e^2\Pi^{5;0}_{3,0} + e'^2\Pi^{3;2}_{3,0} + e^4\Pi^{7;0}_{3,0} + e^2e'^2\Pi^{5;2}_{3,0} + e'^4\Pi^{3;4}_{3,0} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 3g) \\
+ e^2e' \quad & \{\Pi^{2;1}_{2,1} + e^2\Pi^{4;1}_{2,1} + e'^2\Pi^{2;3}_{2,1} + e^4\Pi^{6;1}_{2,1} + e^2e'^2\Pi^{4;3}_{2,1} + e'^4\Pi^{2;5}_{2,1} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g' + 2g) \\
+ ee'^2 \quad & \{\Pi^{1;2}_{1,2} + e^2\Pi^{3;2}_{1,2} + e'^2\Pi^{1;4}_{1,2} + e^4\Pi^{5;2}_{1,2} + e^2e'^2\Pi^{3;4}_{1,2} + e'^4\Pi^{1;6}_{1,2} + \text{etc.}\}^{(i)} A^i \cos(i\lambda' - i\lambda + 2g' + g) \\
+ e^{3} \quad & \{\Pi^{0;3}_{0,3} + e^2\Pi^{2;3}_{0,3} + e'^2\Pi^{0;5}_{0,3} + e^4\Pi^{4;3}_{0,3} + e^2e'^2\Pi^{2;5}_{0,3} + e'^4\Pi^{0;7}_{0,3} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 3g') \\
\\
+ e^4 \quad & \{\Pi^{4;0}_{4,0} + e^2\Pi^{6;0}_{4,0} + e'^2\Pi^{4;2}_{4,0} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 4g) \\
+ e^3e' \quad & \{\Pi^{3;1}_{3,1} + e^2\Pi^{5;1}_{3,1} + e'^2\Pi^{3;3}_{3,1} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g' + 3g) \\
+ e^2e'^2 & \{\Pi^{2;2}_{2,2} + e^2\Pi^{4;2}_{2,2} + e'^2\Pi^{2;4}_{2,2} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 2g' + 2g) \\
+ ee'^3 \quad & \{\Pi^{1;3}_{1,3} + e^2\Pi^{3;3}_{1,3} + e'^2\Pi^{1;5}_{1,3} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 3g' + g) \\
+ e'^4 \quad & \{\Pi^{0;4}_{0,4} + e^2\Pi^{2;4}_{0,4} + e'^2\Pi^{0;6}_{0,4} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 4g').
\end{aligned}
$$

This development is not directly comparable with those hitherto executed, because we use the successive logarithmic derivatives D, instead of the derivatives with respect to α, the ratio of the mean distances. The two classes of derivatives are, however, connected by a simple linear relation which makes it easy to pass from one to the other.